Lecture 22. A Gallery of Solution Curves of Linear Systems

In the previous sections, we talked about the method of solving the differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{1}$$

where $\mathbf{A}$ is an $n \times n$ matrix. Note the eigenvalues and eigenvectors of $\mathbf{A}$ plays an essential role in the solution of Eq. (1).

In this section, we give a brief introduction on the geometric understanding of the role that the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ play in the solutions of the system (1). We will consider the special case when $n = 2$.

First, let's review the Eigenvalue Method. Particulary for a 2×2 matrix $\mathbf{A}$.

Constant Coeff. Homogeneous System:

Constant Coeff. Homogeneous: $\dfrac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$

Solution: $\vec{\mathbf{x}} = c_1\vec{\mathbf{x}}_1 + c_2\vec{\mathbf{x}}_2 + \cdots$,
where $\vec{\mathbf{x}}_i$ are fundamental solutions
from eigenvalues & eigenvectors.
The method is described as below.

The Eigenvalue Method for Homogeneous Systems:

The number λ is called an *eigenvalue* of the matrix $\mathbf{A}$ if $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

An *eigenvector* associated with the eigenvalue λ is a nonzero vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

We consider $\mathbf{A}$ to be 2×2, then the general solution is $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$, with the fundamental solutions $\vec{\mathbf{x}}_1(t), \vec{\mathbf{x}}_2(t)$ found has follows.

- Distinct Real Eigenvalues. $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t}, \vec{\mathbf{x}}_2(t) = \vec{\mathbf{v}}_2 e^{\lambda_2 t}$
- Complex Eigenvalues. $\lambda_{1,2} = p \pm qi$. (*suggestion: use an example to remember the method*)

  If $\vec{v} = \vec{a} + i\vec{b}$ is an eigenvector associated with $\lambda = p + qi$, then

  $\vec{\mathbf{x}}_1(t) = e^{pt}\left(\vec{a}\cos qt - \vec{b}\sin qt\right)$, $\vec{\mathbf{x}}_2(t) = e^{pt}(\vec{b}\cos qt + \vec{a}\sin qt)$
- Defective Eigenvalue with multiplicity 2.
  Find nonzero $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ such that $(\mathbf{A} - \lambda\mathbf{I})^2\vec{\mathbf{v}}_2 = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$.
  Then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda t}$, $\vec{\mathbf{x}}_2(t) = (\vec{\mathbf{v}}_1 t + \vec{\mathbf{v}}_2)\, e^{\lambda t}$.

Let’s consider the differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

where $\mathbf{A}$ is a 2×2 matrix.

To understand the geometric intepretation of the solutions, we consider the following cases:

Real Eigenvalues

We will divide the case where λ_1 and λ_2 are real into the following possibilities:

- Distinct eigenvalues
  - Nonzero and of opposite sign $(\lambda_1 < 0 < \lambda_2)$
  - Both negative $(\lambda_1 < \lambda_2 < 0)$
  - Both positive $(0 < \lambda_2 < \lambda_1)$
  - One zero and one negative $(\lambda_1 < \lambda_2 = 0)$
  - One zero and one positive $(0 = \lambda_2 < \lambda_1)$
- Repeated eigenvalue
  - Positive $(\lambda_1 = \lambda_2 > 0)$
  - Negative $(\lambda_1 = \lambda_2 < 0)$
  - Zero $(\lambda_1 = \lambda_2 = 0)$

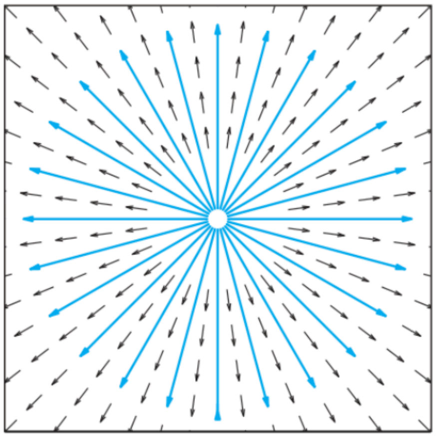
Complex Eigenvalues

In this case, we have $\lambda = p \pm iq$

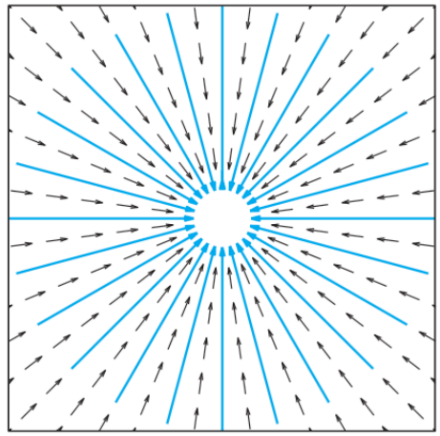
- Purely imaginary ($\operatorname{Re} \lambda = 0$)
- Positive real part ($\operatorname{Re} \lambda > 0$)
- Negative real part ($\operatorname{Re} \lambda < 0$)

📌 The next two pages summarizes the gallery of typical phase plane portraits for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Explaining every phase plane in detail would take a week of classes, so we won't go into too much depth. Instead, I'll show you some examples of specific cases.

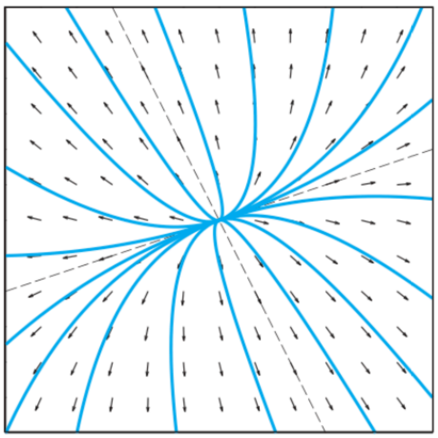
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = \mathbf{A}\mathbf{x}$: Nodes

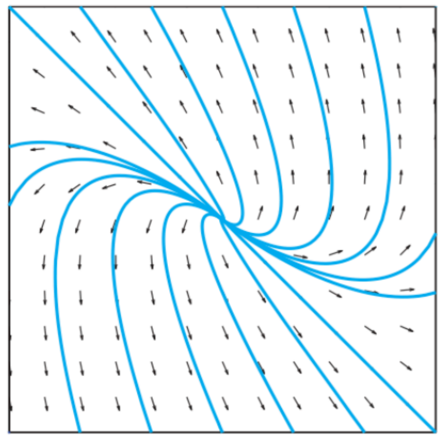


Proper Nodal Source: A repeated positive real eigenvalue with two linearly independent eigenvectors.

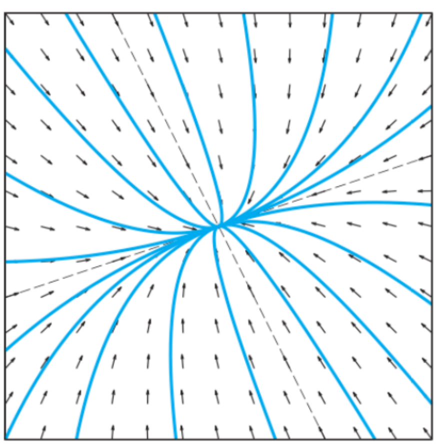


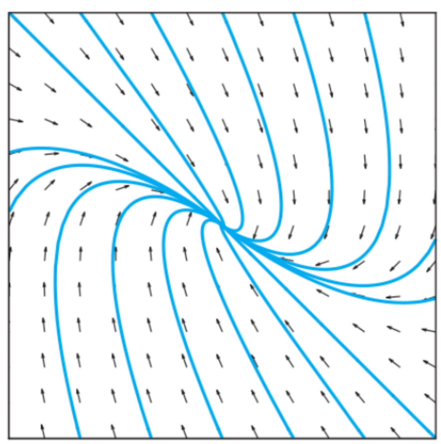
Proper Nodal Sink: A repeated negative real eigenvalue with two linearly independent eigenvectors.





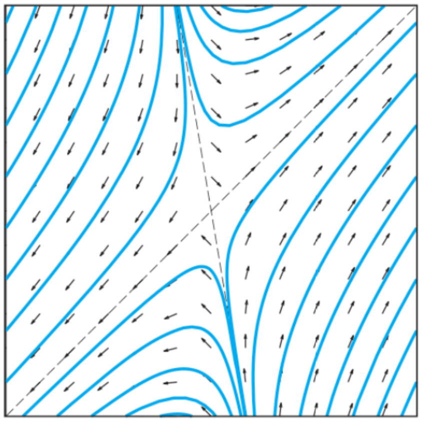
Improper Nodal Source: Distinct positive real eigenvalues (left) or a repeated positive real eigenvalue without two linearly independent eigenvectors (right).



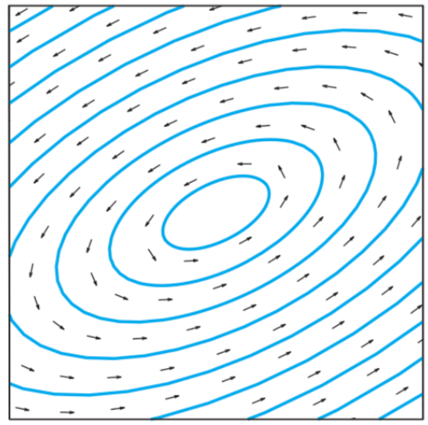


Improper Nodal Sink: Distinct negative real eigenvalues (left) or a repeated negative real eigenvalue without two linearly independent eigenvectors (right).

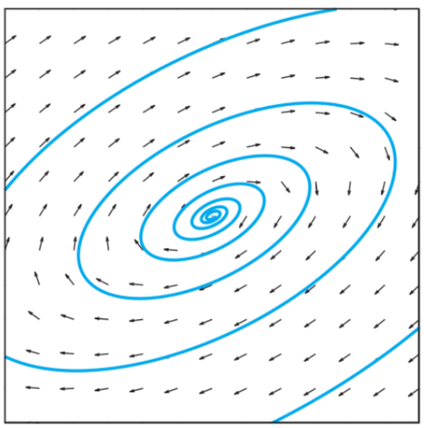
Gallery of Typical Phase Portraits for the System $\mathbf{x}' = A\mathbf{x}$: Saddles, Centers, Spirals, and Parallel Lines



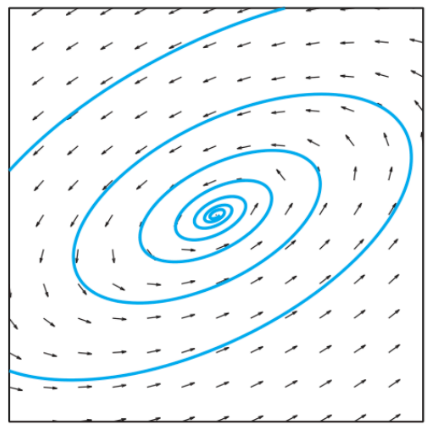
Saddle Point: Real eigenvalues of opposite sign.



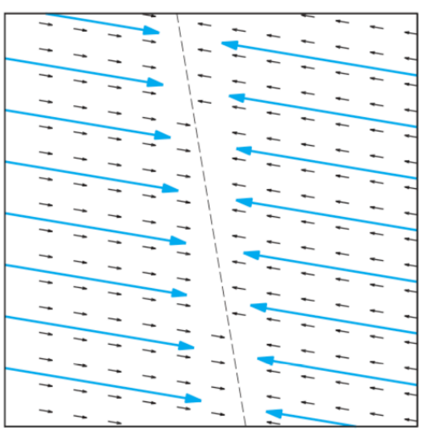
Center: Pure imaginary eigenvalues.



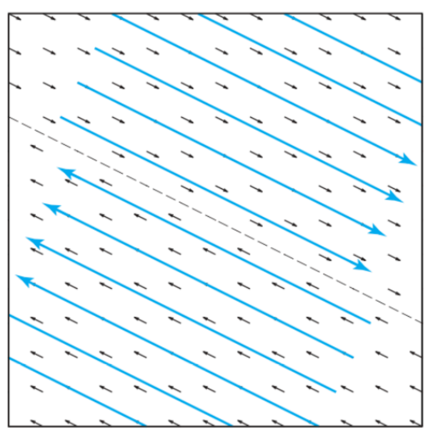
Spiral Source: Complex conjugate eigenvalues with positive real part.



Spiral Sink: Complex conjugate eigenvalues with negative real part.



Parallel Lines: One zero and one negative real eigenvalue. (If the nonzero eigenvalue is positive, then the trajectories flow *away* from the dotted line.)

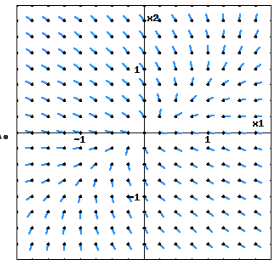
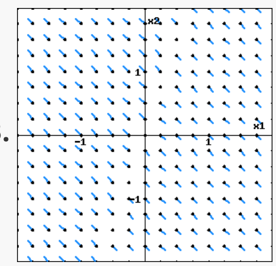
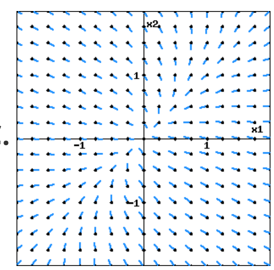
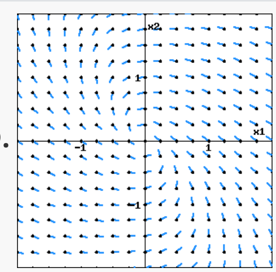


Parallel Lines: A repeated zero eigenvalue without two linearly independent eigenvectors.

Example 1.

Match each linear system with one of the phase plane direction fields. (The blue lines are the arrow shafts, and the black dots are the arrow tips.)

Note: To solve this problem, you only need to compute eigenvalues. In fact, it is enough to just compute whether the eigenvalues are real or complex and positive or negative.

Linear Systems	Phase Plane
1. $\mathbf{x}' = \begin{bmatrix} 7 & 5 \\ -5 & 1 \end{bmatrix} \mathbf{x}$	**A.** 
2. $\mathbf{x}' = \begin{bmatrix} 6 & -3 \\ -1 & 4 \end{bmatrix} \mathbf{x}$	**B.** 
3. $\mathbf{x}' = \begin{bmatrix} -5 & 4 \\ 1 & -5 \end{bmatrix} \mathbf{x}$	**C.** 
4. $\mathbf{x}' = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{x}$	**D.** 

1. Compute the eigenvalues for $A = \begin{bmatrix} 7 & 5 \\ -5 & 1 \end{bmatrix}$, we find

$\lambda = 4 \pm 4i$

Thus for 1, we know A has complex-valued eigenvalues with with real part of $\lambda > 0$. From the Gallery of Phase Plane, we know the origin is a spiral source. So the correct answer is D.

Further discussion, if we completely solve the system, we know

$$\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{4t} \left[\cos 4t \begin{bmatrix} -3 \\ 5 \end{bmatrix} + \sin 4t \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right] + c_2 e^{4t} \left[\cos 4t \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin 4t \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right]$$

Note $t \to \infty$, we have $x(t) \to \infty$ and $y(t) \to \infty$.

2. Find the eigenvalues for $\begin{bmatrix} 6 & -3 \\ -1 & 4 \end{bmatrix}$: $\lambda = 7, \lambda = 3$.

Thus for 2, we know A has distinct positive real eigenvalues. From the Gallery of Phase Plane, we know the origin is an improper Nodal Source.

So the correct answer is C.

3. Find the eigenvalues for $\begin{bmatrix} -5 & 4 \\ 1 & -5 \end{bmatrix}$: $\lambda = -3, \lambda = -7$.

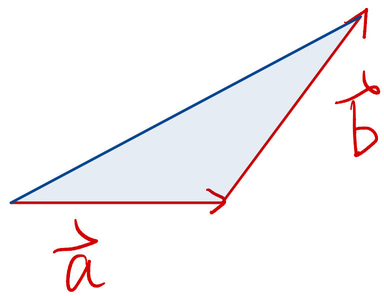
In this case the origin is an improper nodal sink.

So the correct answer is A

4. Find the eigenvalues for $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$: $\lambda = 0, \lambda = -4$

In this case, we know A has one zero and one negative real eigenvalue. From the Gallery of Phase Plane, we know the system has straight lines as solutions

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{a}} + \underbrace{c_2 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}}$$



Example 2.

(1) Find the most general real-valued solution to the linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} -13 & 12 \\ -9 & 8 \end{bmatrix} \mathbf{x}$$

(2) In the phase plane, this system is best described as a

source / unstable node
sink / stable node √
saddle
center point / ellipses
spiral source
spiral sink
none of these

Answer.

(1) Apply the usual eigenvalue and eigenvector method, we find

$$\lambda_1 = -4, \ \mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\lambda_2 = -1, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 4e^{-4t} \\ 3e^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

(2)